Robust solutions and risk measures for a Supply chain planning problem under uncertainty

Chandra A. Poojari Cormac Lucas and Gautam Mitra

Centre for the analysis of Risk and optimisation modelling applications (CARISMA), School of Information Systems, Computing and Mathematics, Brunel University,London

Abstract

We consider a strategic supply chain planning problem formulated as a two-stage Stochastic Integer Programming (SIP) model. The strategic decisions include site locations, choices of production, packing and distribution lines, and the capacity increment or decrement policies. The SIP model provides a practical representation of real world discrete resource allocation problems in the presence of future uncertainties which arise due to changes in the business and economic environment. Such models that consider the future scenarios (along with their respective probabilities) not only identify optimal plans for each scenario, but also determine a hedged strategy for all the scenarios. We,

(1) exploit the natural decomposable structure of the SIP problem through Benders' decomposition,

(2) approximate the probability distribution of the random variables using the Generalised Lambda distribution, and

(3) through simulations, calculate the performance statistics and the risk measures for the two models, namely the expected-value and the here-and-now.

Key words: Supply Chain planning, Stochastic integer Programming, Benders' decomposition, Generalised Lambda distribution, simulation, Genetic algorithm

1 An overview of supply chain decisions under uncertainty

Supply Chain planning is a complex process whereby raw material procurement is followed by the manufacturing process and then finally the finished goods are distributed to the customers. The planning has to cope with short term issues (operational planning such as the daily scheduling), medium term (tactical planning such as the monthly scheduling) as well as long term (strategic planning such as the capacity expansion or reduction). The area is a vibrant research field with a number of books devoted to this topic (Christopher, 1992; Shapiro, 2000). Much of the early research focused on solution algorithms such as the work of (Geoffrion and Graves, 1974) which used a Benders' decomposition approach to solve a single period multi-commodity production distribution problem. More recent work has focused both on more detailed model specifications and specialist algorithms to solve these more complex problems. (Geoffrion and Powers, 1995) describe how the industrial requirements for more detail in the model specification is still driving the research into developing algorithms to solve the resulting problems. (Arntzen et al., 1995) presented a global supply chain problem that minimises cost and delivery time for Digital Equipment Corporation whilst meeting demand in a multi-period multi-commodity setting. (Vidal and Goetschalckx, 1997) review strategic models for planning global distribution and planning problems for supply chain planning and conclude that they are lacking in detail and more research is needed to fill this gap. (Verter and Dasci, 2002) developed a model for determining location of facilities and capacity configurations to be installed at each facility location. Conventionally, the managers in a manufacturing organisation, ignore the interaction of the capacity and the inventory decisions by addressing them separately. In (Bradley and Arntzen, 1999), the authors develop a model that simultaneously plans capacity investment, inventory investment, and the production schedule while maximising the return on assets. They model and analyse two cases, an office supplier and an electronic manufacturer, both involving strategic and tactical decisions.

Whilst much of the earlier research work has looked at describing and solving a deterministic representation of the problem, more recently researchers have turned their attention to the modelling of uncertainty and the corresponding exposure of the risk thereby. Uncertainties can arise due to advances in technology, movements in foreign exchange rates, changes in international taxation schemes, and resource availability (in particular staff). The capturing of such risk further complicates the modelling of supply chain problems. (Huchzermeier and Cohen, 1996) develop a model that uses stochastic exchange rates and maximises after tax profit while providing enough capacity to meet demand. In combining financing issues with decision making, supply chain planning now takes advantage of developments in mathematical finance. A number of strategic planning models use the techniques of option pricing and introduce a real options approach to capacity planning (Nembhard et al., 2000). Similarly, (Haksoz and Seshadri, 2006) consider incorporating of forward contracts for raw material deliveries to hedge against future raw material spot prices. They illustrate how the manufacturer's role can be changed to that of a speculator on movements in the future price of raw materials. From a management perspective, researchers have been investigating uncertainty and the modelling and control of risk. (Christopher and Lee, 2004) describe how the supply chain risk has increased driven by the changes in the business models used for modern supply chain planning. In adopting lean processes, these result in more outsourcing while relying on a much smaller supplier base. Their approach to improve this risk exposure is to encourage all the constituents in the supply chain process to provide more and better quality information. (Brindley and Ritchie, 2001) take this further and propose that relationship building and partnering is the most effective approach to risk management in the absence of a more quantitative approach. Their recent work, (Ritchie and Brindley, 2006) proposes a framework for risk management. First risk drivers are determined, then by following a sequence of procedures the company can determine their actions in risk mitigation. Applying the process within a manufacturing company they found that it resulted in increased sharing of information, building of relationships and the participation of all partners in the risk management process.

Quantitative models continue to play an important role. Scenario based planning and analysis are particularly useful in times of a major structural change in capital intensive industries with long planning horizons, such as oil companies (Dempster et al., 2000), vehicle manufacturers (Eppen et al., 1989), and electricity suppliers (Robinson, 1988). They are built from a realistic combination of key driver values, such as interest-rates, inflation, demand for the products, fluctuation in prices, which are elaborated into fully-fledged narratives by enriching them with information about dependent variables, certain events and the interactions between the many scenario elements. Several scenarios may need to be constructed to provide an insight into the risks, robustness and flexibility of various decisions and a defensible position from which to commit resources. Important issues in scenario generation are (i) *comprehensibility*, that is, it should capture all aspects, both extreme and normal instances, of the underlying distributions and (ii)*consistency*, that is, it must capture correlations among the stochastic data as well. Applications of scenario generation are found in the financial sector (see Carino et al., 1994; Mulvey, 1996; Dempster and Consigli, 1998). (Hoyland and Wallace, 1996) proposed an iterative algorithm that combines simulation, Cholesky decomposition and various transformations to generate scenarios for a Nordic asset management firm.

A mathematical programming problem in which some of the data are unknown, that is, they are subject to uncertainty, random influences, or statistical variations is called a Stochastic Programming (SP) problem. SP provides a general framework to model path dependence of the stochastic process within an optimisation model. Furthermore, it permits uncountably many states and actions, together with constraints, time-lags etc. Unlike dynamic programming, SP separates the model formulation activity from the solution algorithm. One advantage of this separation is that it is not necessary for SP models to obey the same mathematical assumptions. This leads to a rich class of models for which a variety of algorithms can be developed. SP formulations, however, can lead to very large scale problems. This requires the development of efficient solution methods in order to process progressively larger models.

2 Outline of the Paper

Our approach in this paper is a contribution towards business analytics in supply chain. We formulate a practical strategic supply chain model as a two-stage SP problem. We process the ex-ante here-and-now decision problem using Benders' decomposition. Ex-poste analysis is carried out in order to evaluate the performances of the here-and-now and the expected-value decisions. In the absence of knowledge of the probability distribution of the demand, we construct an approximate probability distribution using the generalised lambda distribution. The parameters of the generalised lambda distribution are estimated by processing a non-linear optimisation problem using a Genetic Algorithm. The performance of the decisions are evaluated on the scenarios generated using the generalised lambda distribution. Currently, researchers either use Stochastic Programming or Simulation in order to model Supply chain problems having uncertain parameter values. Our two-faceted approach that combines the Stochastic Programming modelling and Simulation is the main contribution in this paper.

In section 3 we discuss the use of SP for modelling risk-based decision making in Supply chain. In section 4, we formulate a strategic supply chain planning problem having uncertain demand as a Stochastic Integer Programming problem. The resulting large-scale optimisation problem is processed using Benders' decomposition which is detailed in section 5. In section 6 we show how the generalised lambda distribution is used in order to construct an approximate probability distribution of the uncertain demand. The parameters in the generalised lambda are estimated by processing a non-linear optimisation problem using a Genetic Algorithm, and the correlation among the demand for different time period, products and customer zones is captured using Kendall's Tau measure. We then construct a simulation model and for the alternative decisions compute the Value-at-Risk and the Conditional-Value-at-Risk. Finally in section 7, we summarise the results of our investigations.

3 Stochastic Programming for modelling of risk in Supply chain

Location of production assets and the expansion or reduction of their capacities form a crucial aspect of many strategic planning applications. Examples of such applications can be found in heavy process industries(Manne, 1967), communication networks (Chang and Gavish, 1993) (Laguna, 1998), electric utilities (Murphy et al., 1987) (Murphy and H.J.Weiss, 1990), automobile industries (Eppen et al., 1989), service industries (Berman and Ganz, 1994) (Berman et al., 1994) and in electronic goods and semiconductor industries (Berman and Hood, 1999) (Rajagopalan et al., 1998) (Swaminathan, 2000). In all of these applications, the expansion of production capacity requires the commitment of substantial capital resources over long periods of time. Furthermore, the economies of scale in the expansion costs, as well as the uncertainties in the long range forecasts of costs and demands, make these decision problems very complex. Consequently, quantitative models for economic capacity expansion planning has been the subject of intense research since the early 1960s. Conventionally, in the context of Operations Research, mixed integer programming (MIP) models have been formulated for determining strategic decisions. Such models are often inadequate because they completely ignore future uncertainties. A simplified yet pragmatic approach to capture some aspects of these uncertainties is to introduce a set of 'scenarios' representing possible future states of the world. Each scenario is associated with a probability level representing the decision maker's expectation of the occurrence of a particular scenario. This class of models which capture both the optimum resource allocation paradigm and the randomness of the model parameters, are known as Stochastic Programming(SP) problems. SP models with integer variables are known as Stochastic Integer Programs (SIP).

SP models are ideally suited for analysing resource acquisition plans since they explicitly consider randomness and quantify uncertainty governing the key parameters of the model. As a consequence the optimum decisions for strategic plans and tactical operations are more flexible or robust in comparison with decisions obtained by applying deterministic models. SP models have been successfully applied to strategic planning problems for example: electric utility planning (Bienstock and Shapiro, 1988), goods distribution (Cheung and Powell, 1996), capacity planning (Modiano, 1987), (Escudero et al., 1993), (Wagner and Berman, 1995), communication network planning (Fantauzzi et al., 1996), transportation planning and vehicle routing (Fefergruen and Zipkin, 1984), (Laporte et al., 1994).

An important class of stochastic models is the two-stage stochastic linear program with recourse stated as:

$$
\begin{aligned}
Min \;\; & c^T x + E_\xi Q(x,\xi) \\
& \text{subject to} \\
& Ax = b, \\
& x \geq 0,
\end{aligned} \tag{1}
$$

where $Q(x,\xi) = Min\{q^T y | Dy = h - Bx, y \geq 0\}$, ξ is the vector formed by the components of q^T, h, B and D, and E_ξ denotes mathematical expectation with respect to ξ. The vector x represents the *first-stage* (strategic) decisions to be taken without full information on the random variable ξ. Later, complete information is received on the realization of the random vector ξ, then, *second-stage* (operational) or corrective actions y are taken. The expected value of the second-stage cost function $E_\xi Q(x,\xi)$ is referred to as the recourse function (Birge and Louveaux, 1997).

Since the SP model involves submodels describing a set of scenarios of the company's future (uncertain) operations, they usually lead to problems of substantial dimensions. A detailed description of how the hierarchical structure of planning systems can be exploited for building and solving two-stage and multistage stochastic decision models can be found in (Birge and Louveaux, 1997; Dempster et al., 1983; Stougie, 1987; Bitran and Tirupati, 1993).

Whilst a decision analysis (Keeney and H.Raiffa, 1976) approach is well suited to achieving coherence and explicit communication, the skill in this area is to get an acceptable amount of detail into the model for credibility, without it becoming too complicated and confusing. SP being an extension of Linear/Mixed integer programming allows processing of resource allocation models having a number of variables and constraints. As in Linear/Mixed integer programming there is no coupling between the SP modelling and the SP solution algorithms hence sophisticated algorithms can be tuned to exploit the model structure. With the availability of parallel communication protocol real-life models having over a million scenarios, have been processed (Fragniere et al., 2000) on desktop PCs.

SP provides three ways of analysing such large-scale models

- construct a separate strategy for each scenario and test whether or not such strategies are acceptable for the other scenarios;
- construct a solution that hedges against all the scenarios with respect to some metric such as the expected cost;
- construct a solution that satisfies a set of conditions with a given level of reliability.

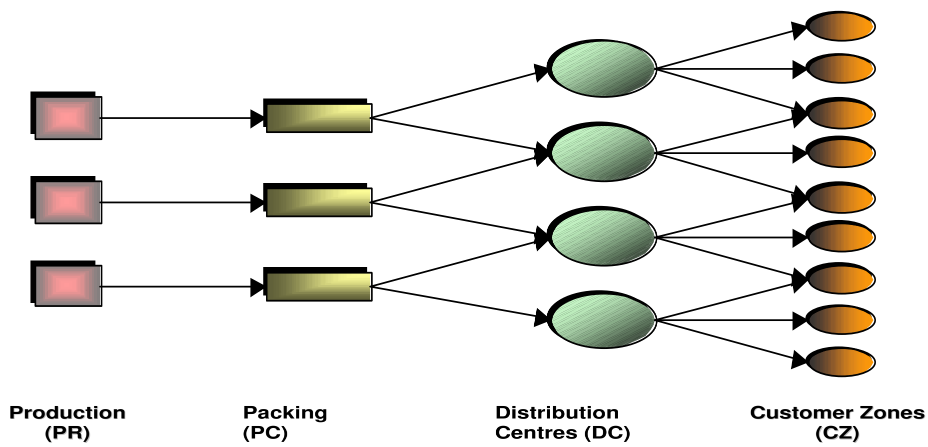


Fig. 1. The supply chain planning network.

4 A supply chain capacity planning model under uncertain demand

Our model represents the entire manufacturing supply chain, from the acquisition of raw material to the delivery of the final products. In order to capture the interactions among different echelons of the manufacturing process at each time period $t \in T$, the network is simplified to four echelons related to production, packing, distribution and customer zones. In figure 1, PR, PC, DC, and CZ denote the sets of production plants, packing plants, distribution centres and customer zones, respectively. The problem is a multi-period multiechelon model representing strategic decisions, such as site locations, choices of production, packing and distribution lines and their capacity increment or decrement policies.

The complete formulation of the model (Baricelli, 1996) can be divided into three main components.

Logical Constraints

These are purely strategic constraints and ensure logical consistencies; a typical constraint group is shown below,

$$\sum_l r_{til} \leq M_i z_{ti} \quad \forall t, i \tag{2}$$

where r_{til} is a binary variable determining if line l is operational at site i time t or not,
M_i is the maximum number of lines allowed at site i, and
z_{ti} is a binary variable indicating if site i is open time t (z_{ti}=1) or not (z_{ti}=0).

Other logical constraints are modelled to determine when sites or lines are closed down or newly opened. These are necessary since costs and penalties are incurred whenever these events occur.

Capacity Constraints

The second major component of the model comprises constraints linking strategic and operational variables. Most of these restrictions concern site capacities. A typical group of production capacity constraints are illustrated below,

$$\sum_p f_{tilp} q_{tilp} \leq e_{til} C_{til} \quad \forall t, i, l \tag{3}$$

where f_{tilp} is the fraction of line capacity l required to produce one unit of product p time t at site i,
q_{tilp} is the amount of product p produced on line technology type l at site i time t,
e_{til} is the efficiency of line type l at site i time t, and
C_{til} is a reporting variable representing the number of lines of technology l at site i time t.

Demand Constraints

The last major component of the model concerns operational constraints. The bulk of these represent material balances. The demand constraints are illustrated below,

$$\mu_{thp} \geq D_{thp} - \sum_k F_{tkhp} \quad \forall t, h, p \tag{4}$$

where μ_{thp} is the shortage quantity of product p at customer zone h time t,
D_{thp} is the demand of product p from customer h time t,

F_{tkhp} is the amount of product p sent to customer h from distribution centre k, time t.

The uncertainty in the demand for the end-products is captured through a set of scenarios. The first stage decisions are strategic (binary and integer) and consequently are not indexed by this new set. Whereas all the operational decisions are now second stage decisions and incorporate the scenario index, s. Thus the number of constraints in the second and third components of the model grow in dimension by this new index set. Taking this into account and for convenience we set out the most general formulation of the problem as a two-stage SIP model, P_{2SIP}.

$$P_{2SIP} \quad Min \quad Z = cx + \sum_s p_s f y_s$$

$$\begin{aligned}
&\text{subject to} && \\
&\qquad Ax = b && \text{Logical Constraints,} \\
&Bx + Dy_s \leq h \quad \forall\ s \in S && \text{Capacity Constraints,} \\
&\qquad Ey_s = d_s \quad \forall\ s \in S && \text{Demand Constraints,}
\end{aligned}$$

where $x = (x_I, x_B), x_I \in Z^{n_1^{'}}, x_B \in \{0,1\}^{n_1^{''}}, y_s \in \Re^{n_2}, y_s \geq 0$ S denotes the set of scenarios, whereby $s = 1, \ldots |S|$
p_s denotes probability of occurrence of scenario s
x denotes the first stage variables
y_s denotes the second stage recourse variables
$b \in R^{m_1}, h \in R^{m_2}, d_s \in R^{m_3}, c \in R^{n_1}, f \in R^{n_2}, A \in R^{m_1 \times n_1}, B \in R^{m_2 \times n_1}, D \in R^{m_2 \times n_2}, E \in R^{n_2 \times n_2}$.

Table 1 summarises a single scenario deterministic mixed integer programming (MIP) model presented as a detailed planning model (in our case there are 100 scenarios).

5 Computational investigation: Processing the SIP decision problem

A distinguishing feature of two and multi-stage SP model is that the dynamics of uncertainty is discretised as a scenario tree in which nodes represent probabilistic states of information, which is supplemented by a linear dynamical system of vectors representing auxillary aspects of state. The structure of the resulting dual-block angular system can be decomposed effectively to compute solutions for progressively larger models in reasonable time. There are two types of decomposition based approaches depending on whether the scenario tree is split according to time stages- primal decomposition or scenarios-

Table 1
The Model Dimensions

Number of Sites, I			8
Types of packing line technology, Y_C			4
Types of production line technology, Y_R			2
Number of distribution centers, J			15
Types of DC line technology, Y_D			2
Number of Customer Zones, H			30
Number of Products, P			13
Number of time periods, T			6
Model Statistics			
Logical Constraints : Sites, DCs opening and closing, Limit on number of sites, DCs and Lines		$m_1 = 968$	
Operational Constraints : Production, Packing, Ordering, Transportation, Balance,	Capacity (Mixed)	$m_2 = 850$	
Demand and Production and Packing Capacities.	Demand (Continuous)	$m_3 = 4950$	6768
Discrete Decision Variables : Sites, DCs, Production lines, Packing lines, DC lines.		$n_1 = 2096$	
Continous Variables : Production, Packing, Ordering, Transportation and Shortage quantities.		$n_2 = 54400$	56496
Non-zeroes		1154034	
Scenarios		100	

dual decomposition. These decomposition based algorithms are iterative procedures generating progressively better solutions. Each iteration consists of solving a master problem and several *independent* subproblems- a reason why these methods are specially suited for parallelisation. The appeal of such techniques lies in the fact that they solve the original problem by iteratively solving a sequence of manageable subproblems.

Benders' decomposition

The L-shaped decomposition (Slyke and Wets, 1969) is derived from the observation that the recourse cost function $Q(x, \xi(\omega))$ are convex and polyhedral (piece-wise) linear. The expected recourse cost $Q(x)(= E_{\xi}Q(x, \xi(\omega)))$ is also convex and polyhedral when the uncertain parameters follow a finite, discrete distribution (Wets, 1974). Based on these observations, the implicit form of $Q(x)$ can be restated in terms of outer linearisation of the nonlinear recourse function. From convexity of $Q(x, \xi(\omega))$, it follows that a linear approximation is a lower support. The method solves an approximation of the two-stage linear program using an outer linearisation of $Q(x)$. Two types of constraints are sequentially added: (i) feasibility cuts determining $\{x|Q(x) < +\infty\}$and (ii) optimality cuts which are linear supports of $Q(x)$ on its domain of finiteness. The basic idea of the L-shaped method is to approximate the nonlinear recourse function in the objective of these problems. A general principle behind this approach is that, since the recourse function involves a solution of all second-stage recourse linear programs, we would like to avoid numerous function evaluations for it. We therefore use that term to build a master problem in x but we evaluate the recourse function exactly as a sub-problem.

The method can be visualised as solving a problem associated with each node of the scenario tree. These problems are formed by the realisation of the associated random parameters. The tree is traversed forwards and backwards, with information from the solution to each nodal LP problem being passed to its immediate descendants by the formation of their right hand side and to its immediate ancestors in the form of cuts. This method has also been generalised to multiple stages (Birge, 1985).

Our computations were performed on Windows 2000 having 2.4 GHz and 2 MB RAM. We used CPLEX (ILOG, 2005) version 9 as the solver subroutine. We investigate the solutions to three formulations of the supply chain problem. First, we consider the *Expected Value* model in which the stochastic demands are replaced by the expected value. Second, we consider the *Wait-and-See* problems for each scenario, this corresponds to analysing each scenario separately. Finally, we compute the *hedged* solution by solving the *Here-and-Now* problem. Table 2 shows the objective values and the time taken in order to process the Expected Value, Here-and-Now, and the Wait-and-See problems.

Model Type	**Objective**	**Time**(s)
Expected Value	1.01483×10^7	7245
Here-and-Now	1.14982×10^7	58258
Wait-and-See	1.01462×10^7	67369

Table 2
Solution to the Expected Value, Here-and-Now, and the Wait-and-See problems.

6 Simulation and Back testing

6.1 Experimental framework

SP provides a framework for a hedged strategy (decisions) whereas simulation provides a framework for evaluating such a strategy. By combining SP and simulation we bring together the two frameworks which contribute towards the problem owner's insight into the model. Quantitative researchers and theoreticians have been pushing to integrate stress tests within the general risk management framework. In part this has taken the form of exploring mathematical techniques such as extreme value theory - a statistical approach to improve estimates of the *tails* or extremes of distributions- to see whether rare events can be treated in a way that is more rigorous, and more tractable.
Stress tests is a mix of quantitative technique, expert judgement, a behavioral approach and market intuition. This is particularly true in terms of specifying how fundamental risk factors interact with one another in a stressed market: known sensitivities and scenarios have to be layered into one another and made to behave in an economically plausible way, using expert judgement.

In order to test the robustness and the risk measure of the expected-value and the here-and-now decisions, we evaluate them using out-of-sample scenarios. Such out-of-samples could have been generated easily had we known the underlying probability distribution of the demand. In our case, however, the initial (100) demand scenarios were generated by the domain expert through a combination of subjective and objective techniques.

The demand vectors are indexed over the different products, customers and time-periods. Therefore, the assumption of independence amongst the different components of the demand is not appropriate. In case the marginal probability distribution of the components are known, then multi-variate copulas could be constructed. In the absence of closed form representation for the marginal distribution of the demand and the lack of knowledge about the dependence between the components, it is not possible to use many of the established statistical techniques in order to construct the multivariate joint distribution of the demand vector. Therefore our approach is to first approximate the marginal distributions using the Generalised Lambda Distribution (GLD) and then to combine the marginal distribution using the correlation structure amongst the components. In algorithm 6.1 we describe our pseudocode for simulation. In this pseudocode, we compute the Value-at-Risk and the Conditional-Value-at-Risk for the expected-value and the here-and-now decisions.

Algorithm 6.1: $\text{GLDANDSIMULATION}(P_{SIP}, \Xi, x^*_{EV}, x^*_{HN})$

$N \leftarrow$ Number of out-of-sample scenarios

comment: Compute the moments and the correlation matrix

for all $\xi_1, \xi_2, \ldots \xi_{|S|} \in \Xi$, let $|\xi_i| = K$

for $j \leftarrow 1$ **to** K

do $\begin{cases} \text{Calculate the mean} \mu_j \\ \text{Calculate the variance} \sigma_j^2 \\ \text{Calculate the skewness} \alpha_j^3 \\ \text{Calculate the kurtosis} \alpha_j^4 \end{cases}$

comment: Compute the correlation matrix

Calculate the Kendall's Tau correlation matrix $\Xi_\rho (\in \Re^{K \times K})$

for $j \leftarrow 1$ **to** K

do $\begin{cases} \text{Estimate the parameters } \lambda_1^j, \lambda_2^j, \lambda_3^j, \lambda_4^j \text{ for the GLD} \\ \text{Construct the GLD } GLD(\lambda_1^j, \lambda_2^j, \lambda_3^j, \lambda_4^j) \end{cases}$

comment: Perform simulation and compute the VAR and CVAR

for $i \leftarrow 1$ **to** N

do
- **for** $j \leftarrow 1$ **to** K
  - **do** Sample $\hat{\xi}^j$ from $GLD(\lambda_1^j, \lambda_2^j, \lambda_3^j, \lambda_4^j)$
- Define $\hat{\xi} = \{\hat{\xi}^1, \hat{\xi}^1, \ldots, \hat{\xi}^K\}$
- **comment:** Capture the correlation amongst the demand vectors
- Calculate $\eta = \Xi_\rho \times \hat{\xi}$
- Fix the first-stage solutions x to x^*_{EV}/x^*_{HN}
- Fix the demand vector to η
- Solve the resulting optimisation problem and store the objective function value

Calculate the Value-at-Risk and the Conditional-Value-at-Risk

output $(VAR_{EV}, CVAR_{EV}, VAR_{HN}, CVAR_{HN})$

The inverse problem of finding the probability distribution from a given moment sequence is a hard problem as shown first by Stieltjes. The Generalised Lambda Distribution (Ramberg and Schmeiser, 1974) is a four parameter generalization of Tukey's Lambda family (Hastings et al., 1947), that has proved useful in a number of different applications. GLD can assume a wide variety of shape, and offers risk mangers flexibility in modelling a broad range of probability distributions. Due to its versatility, however, obtaining approximate parameters for the GLD is very challenging (Karian and Dudewicz, 2000). One of the methods for estimating the parameters of GLD is based on matching the first four moments of the empirical data. Tukey's λ family of distribution is defined by the quantile function *Q(u)*.

$$Q(u) = \begin{cases} \frac{u^\lambda - (1-u)^\lambda}{\lambda} & \lambda \neq 0 \\ \frac{log(u)}{1-u} & \lambda = 0 \end{cases} \tag{5}$$

A four parameter generalization of equation 5 was proposed by Ramberg and Schmeiser. For this generalisation, the quantile function is given by

$$Q(u) = \lambda_1 + \frac{u^{\lambda_3} - (1-u)^{\lambda_4}}{\lambda_2} \tag{6}$$

(Ramberg et al., 1979) note that the proposed distribution in equation 6 is not defined for certain combinations of the parameters. Fremier et al. (Freimer et al., 1988) devise a different parametrization for the GLD, denoted as FMKL

$$Q(u) = \left(\lambda_1 + \frac{1}{\lambda_2}(\frac{u^{\lambda_3 - 1}}{\lambda_3} - \frac{(1-u)^{\lambda_4} - 1}{\lambda_4}\right) \tag{7}$$

This parametrization is well defined over the entire two-dimensional plane for the parameter λ_3 and λ_4. However, in order to have a finite k^{th} order moment it is necessary that $min(\lambda_3, \lambda_4) >= -1/k$. Using the relationship *Q(u)* $=$ *x* and *F(x)* =u, we get *f(x)*= *f(Q(u))*. Since *f(x)* $= \frac{dF}{dx}$, therefore *f(x)* $= \frac{du}{dx} = \frac{du}{dQ(u)} = \frac{1}{\frac{dQ(u)}{du}} = \frac{\lambda_2}{u^{\lambda_3 - 1} + (1-u)^{\lambda_4 - 1}}$.

6.3 An optimisation problem for matching moments

Given a GLD with quantile function *Q(u)*, we need to find the parameters $\lambda_1, \lambda_2, \lambda_3$ and λ_4 such that the mean (μ), variance (σ^2), skewness (α^3) and kurtosis (α^4) of the GLD match the corresponding mean ($\hat{\mu}$), variance ($\hat{\sigma}^2$), skewness ($\hat{\alpha}^3$) and kurtosis ($\hat{\alpha}^4$) of the sample. The FMKL parameter for the quantile function can be rewritten as

$$F^{-1}(u) = \left(\lambda_1 - \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_2\lambda_4} + \frac{1}{\lambda_2}\left(\frac{u^{\lambda_3}}{\lambda_3} - \frac{(1-u)^{\lambda_4}}{\lambda_4}\right)\right) \tag{8}$$

$P(u) = \frac{u^{\lambda}_3}{\lambda_3} - \frac{(1-u)^{\lambda_4}}{\lambda_4}$, q_k = the k^{th} central moment of $F^{-1}(u)$, and p_k = the k^{th} raw moment of $P(u)$.

$$q_1 = \lambda_1 - \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_2\lambda_4} + \frac{p_1}{\lambda_2} \tag{9}$$

$$q_2 = \frac{1}{\lambda_2^2}(p_2 - p_1^2) \tag{10}$$

$$q_3 = \frac{1}{\lambda_2^3}(p_3 - 3p_1p_2 + 2p_1^3) \tag{11}$$

$$q_4 = \frac{1}{\lambda_2^4}(p_4 - 4p_1p_3 + 6p_1^2p_2 - 3p_1^4) \tag{12}$$

where $p_k = \int_0^1 \left(\frac{u^{\lambda_3}}{\lambda_3} - \frac{(1-u)^{\lambda_4}}{\lambda_4}\right)^k du$. Using Binomial expansion on p_k, we obtain

$$\begin{aligned} p_k &= \int_0^1 \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} (-1)^j \frac{u^{\lambda_3(k-j)}}{\lambda_3^{k-j}} \frac{(1-u)^{\lambda_4 j}}{\lambda_4^j} du \\ &= \sum_{j=0}^{k} \frac{(-1)^j}{\lambda_3^{k-j}\lambda_4^j} \begin{pmatrix} k \\ j \end{pmatrix} \beta(\lambda_3(k-j)+1, \lambda_4 j + 1) \end{aligned} \tag{13}$$

where $\beta(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}du$. Since the beta function is defined only for the positive values of a and b, we have

$$min(\lambda_3, \lambda_4) > -\frac{1}{4}.$$

The first four raw moments of the distribution are given as follows:

$$
\begin{aligned}
p_1 &= \frac{1}{\lambda_3(\lambda_3+1)} - \frac{1}{\lambda_4(\lambda_4+1)} && (14)\\
p_2 &= \frac{1}{\lambda_3^2(2\lambda_3+1)} - \frac{1}{\lambda_4^2(2\lambda_4+1)} - \frac{2}{\lambda_3\lambda_4}\beta(\lambda_3+1,\lambda_4+1) && (15)\\
p_3 &= \frac{1}{\lambda_3^3(3\lambda_3+1)} - \frac{1}{\lambda_4^3(3\lambda_4+1)} - \frac{3}{\lambda_3^2\lambda_4}\beta(2\lambda_3+1,\lambda_4+1)\\
&\quad + \frac{3}{\lambda_3\lambda_4^2}\beta(\lambda_3+1,2\lambda_4+1) && (16)\\
p_4 &= \frac{1}{\lambda_3^4(4\lambda_3+1)} + \frac{1}{\lambda_4^4(4\lambda_4+1)} + \frac{6}{\lambda_3^2\lambda_4^2}\beta(2\lambda_3+1,2\lambda_4+1)\\
&\quad - \frac{4}{\lambda_3^3\lambda_4}\beta(3\lambda_3+1,\lambda_4+1) - \frac{4}{\lambda_3\lambda_4^3}\beta(\lambda_3+1,3\lambda_4+1) && (17)
\end{aligned}
$$

Substituting the equations 14,15 16, 17 in 9, 10, 11 and 12 we get,

$$
\begin{aligned}
\alpha_3 &= \frac{p_3 - 3p_1p_2 + 2p_1^3}{(p_2 - p_1^2)^{\frac{3}{2}}} && (18)\\
\alpha_4 &= \frac{p_4 - 4p_1p_3 + 6p_1^2p_2 - 3p_1^4}{(p_2 - p_1^2)^4} && (19)
\end{aligned}
$$

The values of the parameters λ_3 and λ_4 of the GLD can be obtained by solving the optimisation problem

$$
\begin{aligned}
&\text{Min} \quad (\alpha_3 - \hat{\alpha}_3)^2 + (\alpha_4 - \hat{\alpha}_4)^2\\
&\text{Subject to the constraints in the equation } 14, 15, 16, 17 && (20)\\
&\text{where } \lambda_3 \in (\frac{-1}{4}, \infty) \& \lambda_4 \in (\frac{-1}{4}, \infty). && (21)
\end{aligned}
$$

Once the values for λ_3 and λ_4 are obtained, then λ_1 and λ_2 can be computed using equations 22 and 23.

$$
\begin{aligned}
\lambda_1 &= \hat{\mu} + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_3+1} - \frac{1}{\lambda_4+1}\right) && (22)\\
\lambda_2 &= \frac{\sqrt{p_2 - p_1^2}}{\hat{\sigma}} && (23)
\end{aligned}
$$

The optimisation problem 20 is non-linear and non-convex. Some of the existing techniques to solve it include Least square (Ozturk and Dale, 1985), Starship method (King and MacGillivray, 1999), Downhill simplex (Nelder and Mead, 1965), Powell's method (Powell, 1962). We develop a *steady-state GA* that uses 'overlapping' populations and a parameter-less penalty function in order to process constrained optimisation problems (see (Poojari and Varghese, 2006) for details). Our GA start out with an initial *population* of possible

Parameter	**Value**
Genetic Algorithm	Steady State
n	dimension of the individual= 2
Size of the population(K)	$10 \times n$
Number of generation	$20 \times$ K
Chromosome representation	real numbers (represented as floating point)
Probability of crossover (p_{cross})	0.7
Probability of mutation (p_{mut})	0.1
Proportion replaced (p_{repl})	0.5

Table 3
Control parameters for the Genetic Algorithm.

solutions to a given problem where each individual is represented using some form of encoding as a *chromosome*. Newly generated offspring are added to the population, then the worst individuals are destroyed. The new offspring may or may not make it into the population, depending on whether they are better than the worst in the population. The individuals in the population are represented as real arrays where the arrays represent the solution vector to the optimisation problem. These chromosomes are evaluated for their *fitness*. The fitness function is designed to provide greater emphasis on optimality during the initial generations of the GA, so as to avoid the algorithm converging to a local optima. Based on their fitness as a criterion, certain chromosomes in the population are selected for reproduction. These selected individuals are manipulated by *crossover* and *mutation* operators. The crossover operator is applied to a pair of selected parents to create offspring, and the mutation operator is used as a slight modification to this offspring, or to the remaining members of the population.

6.4 The Mixing model

The demand vectors are indexed over multiple time-periods, products and customer zones. Through the GLD we approximate the marginal probability distribution of each component of the demand vector. However, these components may be correlated. For instance the demand for a given product in one time-period may influence the demand for the same/different product in another time-period. We use demand scenarios in order to approximate the correlation amongst the components.

It is well-known that the Pearson correlation matrix measures the linear cor-

Number of Sample Size = 1000
Time for constructing the correlation matrix = 335seconds
Time for estimating parameters using GA = 865 seconds
Time for evaluating the *Expected Value* solution = 34000 seconds
Time for evaluating the *Here-and-Now* solution = 48000 seconds

Table 4
Details of the Simulation.

Here-and-Now		**Expected Value**	
VAR	CVAR	VAR	CVAR
2585221.579	2612735.274	2595209.474	2622703.842

Table 5
The 95% values of VAR and CVAR

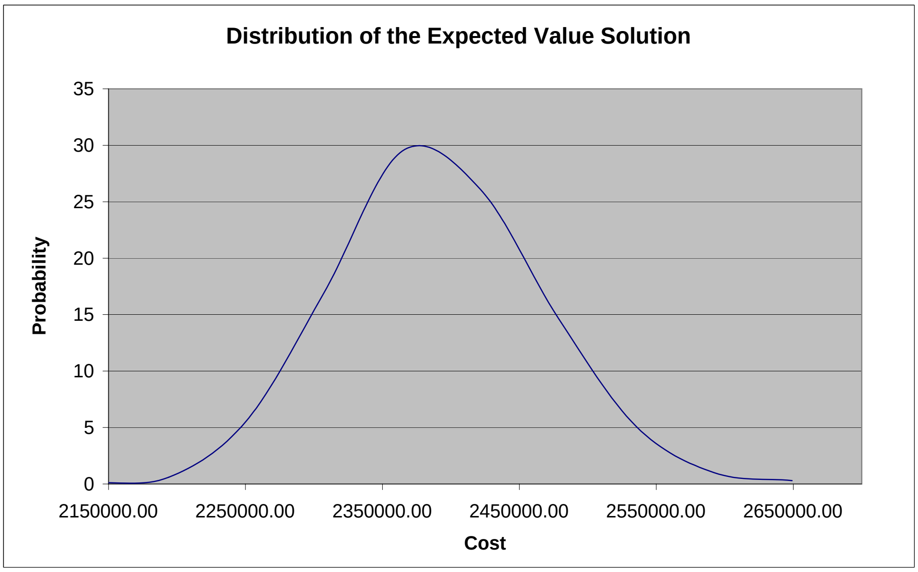
relation, and the rank based correlation coefficients such as the Kendall's tau or the Spearman's rho measure the monotonic association. The rank coefficients are able to detect the correlation when the correlation is nonlinear but monotonic and is also known to be robust is the presence of outliers.

We generate the random vectors (η) using the model

$$\eta = \Xi_\rho \hat{\xi} \tag{24}$$

where $\hat{\xi}^i$ is the i^{th} marginal distribution denoted by the generalised lambda distribution $G(\lambda_1^i, \lambda_2^i, \lambda_3^i, \lambda_4^i)$, and Ξ_ρ is the $m \times m$ Kendall's tau correlation matrix of the original scenarios. The details of the simulation are shown in table 4.

We simulate the performance of the *Here-and-Now* and the *Expected Value* solutions. Theoretically it is known that the *Here-and-now* solution provides a hedged solution and is more robust, therefore it should have lower VAR and CVAR values. Table 5 shows the 95% Value-at-Risk (VAR) and the Conditional-Value-at-Risk (CVAR) values for the the *Expected value* and the *Here-and-Now* solutions. The figure 2 shows the distribution of the cost for the *Expected Value* and the *Here-and-Now* solution respectively.



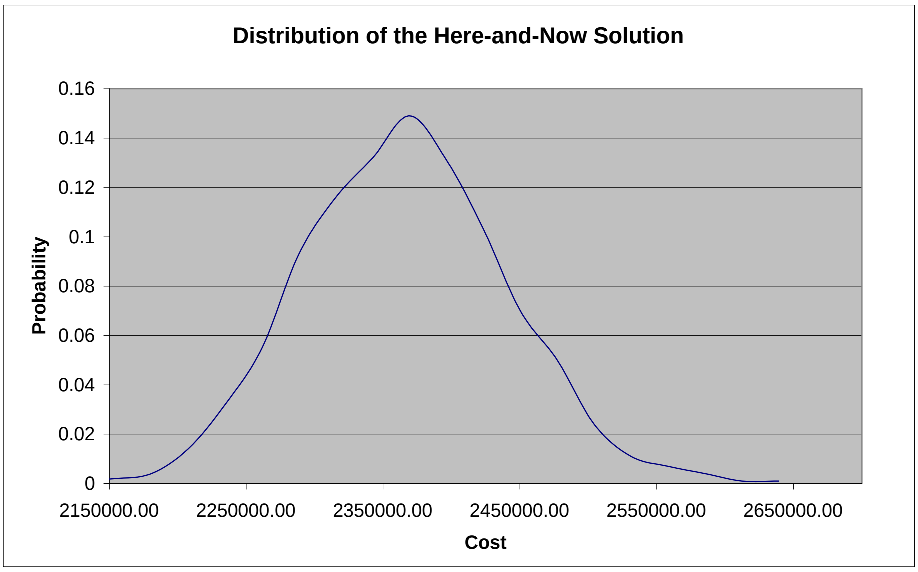


Fig. 2. The distribution of the cost for the Expected value and the Here-and-Now solution.

7 Discussion and Conclusion

We have investigated a strategic capacity planning problem having uncertain demand formulated as a two-stage Stochastic Integer Programming problem.

The underlying optimisation is NP-Hard due to he presence of integer and binary variables. These discrete variables correspond to the strategic decisions such as site locations, choices of production, packing and distribution lines, and the capacity increment or decrement policies. We have implemented Benders' decomposition to process the problem. Further, we verify the robustness of the solution obtained for the stochastic problem through simulation. In order to do this, we have developed a Generalised Lambda Distribution for the demand. The parameters for the Generalised Lambda distribution are estimated by processing a non-linear and non-convex optimisation problem suing a Steady-State Genetic algorithm. We evaluate the ex-ante decisions of the expected-value and the here-and-now optimisation problems through simulation of the random demand. Also, by fixing the first-stage decisions in the underlying optimisation problem, we compute well-known measures of risk such as the Value-at-Risk and the Conditional-Value-at-Risk. We, thus, provide a framework for obtaining robust solutions and computing the risk measures for a practical supply chain planning problem.

References

Arntzen, B., Brown, G., Harrison, T. P., Trafton, L., 1995. Global supply chain management at digital equipment corporation. Interfaces 1 (25), 69–93.

Baricelli, P., 1996. An investigation into a manufacturing and supply chain planning problem under uncertainty. Master's thesis, Department of Mathematics and Statistics, Brunel University, West London.

Berman, O., Ganz, Z., 1994. The capacity expansion problem in the service industry. Computers and Operations Research 21, 557–572.

Berman, O., Ganz, Z., Wagner, J., 1994. A stochastic optimization model for planning capacity expansion in a service industry under uncertain demand. Naval Research Logistics 41, 545–564.

Berman, O., Hood, S., 1999. Capacity optimization planning system (caps). Interfaces 29, 31–50.

Bienstock, D., Shapiro, J., 1988. Optimizing resource acquisition decisions by stochastic programming. Management Science 34, 215–229.

Birge, J., 1985. Decomposition and partitioning methods for multi-stage stochastic linear programs. Operations Research 33, 989–1007.

Birge, J., Louveaux, F., 1997. Introduction to Stochastic Programming. Springer Series in Operations Research, Springer-Verlag, New York.

Bitran, G., Tirupati, D., 1993. Handbooks in Operations Research and Management Science: Logistics of Production and Inventory. North-Holland, Ch. Hierarchial Production Planning.

Bradley, J., Arntzen, B., 1999. The simultaneous planning of production, capacity, and inventory in seasonal demand environments. Operations Research 47 (6), 795–806.

Brindley, C., Ritchie, R., 2001. The information-risk conundrum. Marketing Intelligence and Planning 1 (19), 29–37.

Carino, D., Kent, R., Myers, D., Stacy, C., Sylvanus, M., Turner, A., Watababe, K., Ziemba, W., 1994. Russell-yasuda kasai model: an assest liability model for japanese insurance company using multi-stage stochastic programming. Interfaces , 29–49.

Chang, S.-G., Gavish, B., 1993. Telecommunications network topological design and capacity expansion. Telecommunication Systems 1, 99–131.

Cheung, R., Powell, W., 1996. Models and algorithms for distribution problems with uncertain demands. Transport Science 30 (1), 43–59.

Christopher, M., 1992. Logistics and Supply Chain Management. Pitman Publishing, London.

Christopher, M., Lee, H., 2004. Mitigating supply chain risk through improved confidence. International Journal of Physical Distribution and Logistics Management 4 (35), 388–396.

Dempster, M., Consigli, G., 1998. The calm stochastic programming model for dynamic asset-liability management. In: World Wide Asset and Liability Modelling. Cambridge University Press, pp. 464–500.

Dempster, M., Fisher, M., Jansen, L., Lageweg, B., Lenstra, J., Kan, A., 1983. Analysis of heuristics for stochastic programming- results for heirarchical scheduling problems. Mathematics of Operations Research 8 (4), 525–537.

Dempster, M., Pedrón, N. H., Medova, E., Scott, J., Sembos, A., 2000. Planning logistic operations in the oil industry. Journal of Operational Research Society 51 (11), 1271–1288.

Eppen, G., Martin, R., Schrage, L., 1989. A scenario approach to capacity planning. Operational Research 37 (4), 517–525.

Escudero, L., Kamesam, P., King, A., Wets, R.-J., 1993. Production planning via scenario modelling. Annals of Operations Research 43, 311–335.

Fantauzzi, F., Gaivornski, A., Messina, E., 1996. Lecture notes on economics and mathematical systems: Network optimization. In: Pardalos, M., Hearn, D., Hager, W. W. (Eds.), Decomposition methods for network optimisation problems in the presence of uncertainty. Vol. 450. Gainesville, Florida, pp. 234–248.

Fefergruen, A., Zipkin, P., 1984. A combined vehicle routing and inventory allocation problem. Operations Research 32 (5), 1019–1037.

Fragniere, E., Gondzio, J., Vial, J.-P., 2000. Building and solving large-scale stochastic programs on an affordable distributed computing system. Annals of Operations Research 99 (1), 167–187.

Freimer, M., Mudholkar, G., Kollia, G., Lin, C., 1988. A study of the generalized tukey lambda family. Communications in Statistics - Theory and Methods (17), 3547–3567.

Geoffrion, A., Graves, G., 1974. Multicommodity distribution system design by bender's decomposition. Management Science 20 (5), 822–844.

Geoffrion, A., Powers, R., 1995. Twenty years of strategic distribution systems design: An evolutionary perspective. Interfaces (25), 105–128.

Haksoz, C., Seshadri, S., 2006. Accepted in the Journal of the OR Society (UK), special issue on risk in supply chain.

Hastings, C., Mosteller, F., Tukey, J. W., Winsor, C. P., 1947. Low moments for small samples: a comparative study of order statistics. Annals of Mathematical Statistics 18, 413–426.

Hoyland, K., Wallace, S., 1996. Generating scenario trees for multi-stage decision problems. Management Science 26, 1–13.

Huchzermeier, A., Cohen, M., 1996. Valuing operational flexibility under exchange rate risk. Operations Research 44 (1), 100–113.

ILOG, 2005. Www.ilog.com.

Karian, Z., Dudewicz, E., 2000. Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalized Bootstrap Methods. CRC Press, Boca Raton, FL.

Keeney, R., H.Raiffa, 1976. Decisions with Multiple Objectives: Preferences and Value Trade-offs. Wiley, New York.

King, R. A. R., MacGillivray, H. L., 1999. A starship fitting method for the generalized lambda distribution. Australia and New Zealand Journal of Statistics 41 (3), 353–374.

Laguna, M., 1998. Applying robust optimization to capacity expansion of one location in telecommunications with demand uncertainty. Management Science 44, 101–110.

Laporte, G., Louveaux, F., Hamme, L. V., 1994. Exact solution to a location problem with stochastic demands. Transportation Science 28, 95–103.

Manne, A. (Ed.), 1967. Investments for Capacity Expansion. The MIT Press, Cambridge.

Modiano, E., 1987. Derived demand and capacity planning under uncertainty. Operations Research 35 (2), 185–197.

Mulvey, J., 1996. Generating scenario for the towers perrin investment system. Interfaces 26, 1–13.

Murphy, F., H.J.Weiss, 1990. An approach to modeling electric utility capacity expansion planning. Naval Reserch Logistics 37, 827–845.

Murphy, F., Sen, S., Soyster, A., 1987. Electric utility expansion planning in the presence of existing capacity: A nondifferentiable, convex programming approach. Computers and Operations Research 14, 19–31.

Nelder, J. A., Mead, R., 1965. A simplex method for function minimization. Computer Journal 7, 308313.

Nembhard, H., Shi, L., Park, C., 2000. Real option models for managing manufacturing system changes in the new economy. The Engineering Economist 45 (3), 232–258.

Ozturk, A., Dale, R., 1985. Least squares estimation of the parameters of the generalized lambda distribution. Technometrics 27, 81–85.

Poojari, C. A., Varghese, B., 2006. Genetic algorithm based technique for solving chance constrained problems, accepted for publication in the European Journal of Operational Research.

Powell, M., 1962. An iterative method for finding stationary values of a function of several variables. Computer Journal 5 (2), 147–151.

Rajagopalan, S., Singh, M., Morton, T., 1998. Capacity expansion and replacement in growing markets with uncertain technological breakthroughs. Management Science 44, 12–30.

Ramberg, J., Dudewicz, E., Tadikamalla, P., Mykytka, E., 1979. A probability distribution and its uses in fitting data. Technometrics 21, 201–214.

Ramberg, J., Schmeiser, B., 1974. An approximate method for generating asymmetric random variables. Communications of the ACM 17 (2), 78–82.

Ritchie, B., Brindley, C., 2006. Accepted in the Journal of the OR Society (UK), special issue on risk in supply chain.

Robinson, J. R., 1988. Loaded questions: New approaches to utlity forecasting. Energy policy 16 (1), 58–68.

Shapiro, J., 2000. Modeling the supply chain. Brooks/Cole-Thomson Learning, Pacific Grove, CA.

Slyke, R. V., Wets, R. J.-B., 1969. L-shaped linear programs with application to optimal control and stochastic programming. SIAM Journal on Applied Mathematics 17, 638–663.

Stougie, R., 1987. Design and analysis of algorithms for stochastic integer programming. CWI Tract 37, 48–62.

Swaminathan, J., 2000. Tool capacity planning for semiconductor fabrication facilities under demand uncertainty. European Journal of Operational Research 120, 545–558.

Verter, V., Dasci, A., 2002. The plant location and flexible technology acquisition problem. European Journal of Operational Research 136 (2), 366–382.

Vidal, C., Goetschalckx, M., 1997. Strategic production - distribution models: A critical review with emphasis on global supply chain models. European Journal of Operational Research (98), 1–18.

Wagner, J., Berman, O., 1995. Models for planning capacity expansion of convenience stores under uncertain demand and the value of information. Annals of Operations Research 59, 19–44.

Wets, R. J.-B., 1974. Stochastic programs with fixed recourse: the equivalent deterministic programs. SIAM Review 16, 309–339.